1. An optional hard problem on gauge correlation
   (Optional.) You saw in lecture that Spearman’s rho has
   \[ \rho \to 1 - 6\mathbb{E}[F_X(X_1) - F_Y(Y_1)]^2 = \text{Corr}(F_X(X), F_Y(Y)) \]
   the right hand side is called the gauge correlation.
   Suppose
   \[ (X, Y) \sim \mathcal{N}_2 \left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \]
   We’d like to compute the gauge correlation \( G = \text{Corr}(\Phi(X), \Phi(Y)) \). This is quite difficult, and in fact cannot be done exactly. The best you can do is an approximation. Since \( G(\rho) \) is a differentiable function of \( \rho \), you can write an expansion for \( G \) in powers of \( \rho \) for \( \rho \) close to zero:
   \[ G(\rho) = G(0) + c_1 \rho + c_2 \rho^2 + ... \]
   where of course \( G(0) = 0 \).
   Find \( c_1 \), so as to obtain a first order approximation to \( G \).
   The hint given in class is this: you’re going to end up computing
   \[ \text{Cov}(\Phi(X), \Phi(Y)) = \mathbb{E}\Phi(X)\Phi(Y) - \mathbb{E}\Phi(X)\mathbb{E}\Phi(Y) \]
   For the \( \mathbb{E}\Phi(X)\Phi(Y) \), you can use a rule for computing expectations, known as the law of the iterated expectation, as follows. (Try looking it up!)
   \[ \mathbb{E}\Phi(X)\Phi(Y) = \mathbb{E}\left[ \mathbb{E}[\Phi(X)\Phi(Y)|X] \right] = \mathbb{E}\left[ \Phi(X)\mathbb{E}[\Phi(Y)|X] \right] \]
   In this, you may also use that \( Y|X \) is distributed as \( \mathcal{N}(\rho X, 1 - \rho^2) \).

2. Moments of Spearman’s rho
   Show that \( \mathbb{E}(\rho) = 0 \).
   Optionally you may show that \( \text{Var}(\rho) = \frac{1}{n-1} \). You’ll need the following:
   \[ \text{Var} \sum_i i R_i = \sum_i i^2 \text{Var} R_i + \sum_{i \neq j} i \cdot j \cdot \text{Cov}(R_i, R_j) \]
In this (and you’ll have to show this rigorously) we have $\text{Var} \ R_i = (n^2 - 1)/12$ and $\text{Cov}(R_i, R_j) = (-n - 1)/12$. 