Broad Area of Research:

The broad area of research was reconstruction of 3D objects.

Specific Area:

We are interested in developing fast algorithms for Radial Basis Function (RBF) interpolations in high dimensions.
In general, an RBF is a function of the form
$f(\mathbf{x}) = p(\mathbf{x})+\lambda_i\phi\ (|\mathbf{x}-\mathbf{x}_i|)$
where *p* is a polynomial of low degree and the *basic function* ϕ is a real valued function on $[0,\infty)$, usually unbounded and of noncompact support.

Steps:

To start with, basic implementation of the pointwise RBF problem was carried out to get an understanding of the problem in two dimensions. The basis function used for this case was $\phi(r) = r$ and the polynomial was linear.

The surface representation or reconstruction problem can be expressed as
Given n distinct points $\{(x_i, y_i, z_i)\}^n_{i=1}$ *on a surface M in* $\mathbb{R}^3$*, to find a surface* M_0 *that is a reasonable approximation to M.*

We model the surface implicitly with a function $f(x, y, z)$. If a surface *M* consists of all the points (x, y, z) that satisfy the equation $f(x, y, z) = 0$, then we say that *f* implicitly defines *M*. Describing surfaces implicitly with various functions is a well-known technique.
In order to avoid the trivial solution that *f* is zero everywhere, off-surface points are appended to the input data and are given non-zero values.
This gives a more useful interpolation problem: Find *f* such that
$f(x_i, y_i, z_i) = 0$, i = 1, . . . ,n (on-surface points),
$f(x_i, y_i, z_i) = d_i \neq 0$, i = n+1, . . . ,N (off -surface points).

We choose *f* to be the signed-distance function i.e. the value of *f* at each (x_i, y_i, z_i) is the distance of that point from the surface along the normal, +ve if the point is outside the surface and −ve otherwise.

As suggested in Ref 1, we have appended two off-surface points to each data point. It is not possible to estimate normals everywhere, but the USP of the method lies in the fact that even if the normal can't be determined at a point, we can choose not to include off-surface points there without affecting our solution vastly. It is also important to note that the normal drawn at a point should not intersect the surface at any other point.

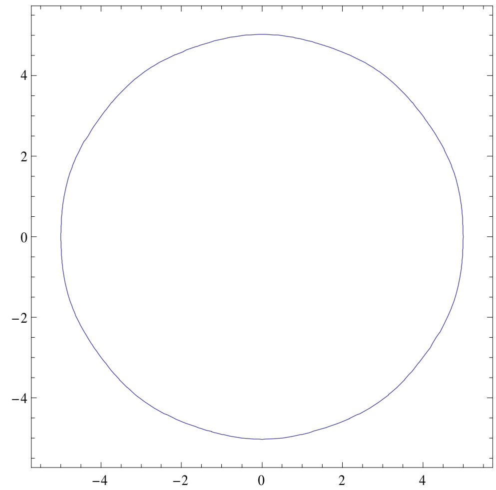
A radial basis function is a function of the distance from the origin. In this case, we use an R.B.F. of the form $f(\mathbf{x}) = p(\mathbf{x})+\Sigma^{N}_{i=1}\lambda_i\phi(||\mathbf{x}-\mathbf{x}_i||)$
which is chosen so as to satisfy the energy-minimisation criterion (Ref 1).
In the above equation, λ_i are real numbers, $||\mathbf{x}_i-\mathbf{x}_j||$ represents the Euclidean distance between the points $\mathbf{x}_i$ and $\mathbf{x}_j$, p is a polynomial of low degree and ϕ (the basic function) is a real-valued function on $[0,\infty)$.
The energy-minimisation criterion also requires that the following conditions hold: $\Sigma_i\lambda_i q(\mathbf{x}_i) = 0$ – (1)
for all polynomials q of degree at most equal to that of p.

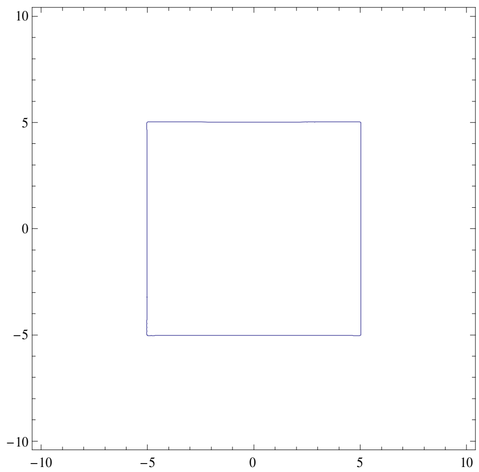
The simplest choice for ϕ is $\phi(r) = r$, for which p is a linear polynomial and the conditions (1) reduce to
$\Sigma_i\lambda_i = \Sigma_i\lambda_i x_i = \Sigma_i\lambda_i y_i = \Sigma_i\lambda_i z_i = 0$ – (1)'

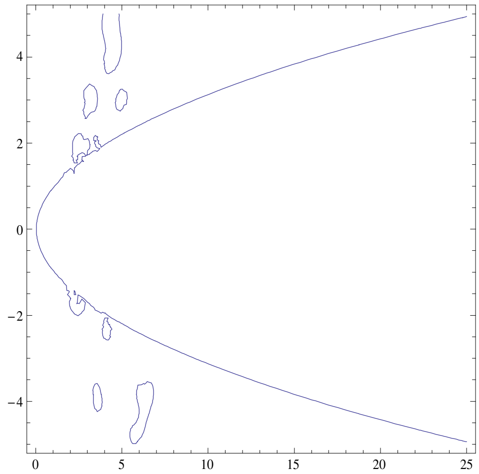
Let $(p_1,p_2,\ldots p_k)$ be a basis for polynomials of degree at most equal to that of p, and let $c= (c_1,c_2,\ldots c_k)'$ be the coefficients that give p in terms of this basis.
Then, the conditions (1), together with the conditions that
$f(\mathbf{x}_i) = f_i$ for i = 1,2,...N
where $f_i = 0$ for $i \le n$
$= d_i$ for $n < i \le N$
give the matrix equation
$\begin{bmatrix} A & P \\ P' & 0 \end{bmatrix} [\lambda, c]' = [f, 0]'$
where $P_{ij} = p_j(\mathbf{x}_i)$ i=1,2,...N and j=1,2,...k
and $A_{ij} = \phi(||\mathbf{x}_i-\mathbf{x}_j||)$ i,j = 1,2,...N
The matrix B = $\begin{bmatrix} A & P \\ P' & 0 \end{bmatrix}$ is symmetric and invertible under very mild conditions on ϕ.
If $\phi(r) = r$, then these relations are modified as
$A_{ij} = ||\mathbf{x}_i - \mathbf{x}_j||$, and ith row of P = $[1, x_i, y_i, z_i]$
In this case, B is invertible if the data-points are not coplanar.

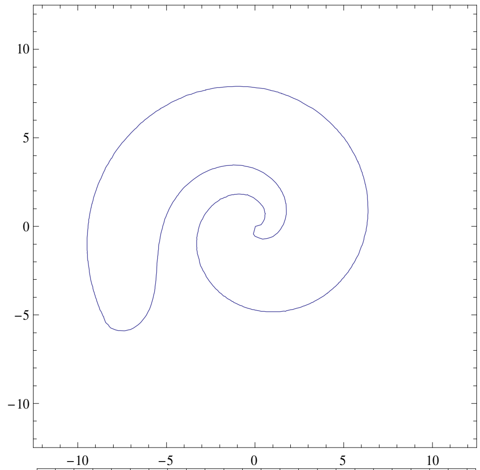
The main challenge facing us now is the inversion of matrix B for $N\sim10^5$ data-points. This is achieved by using Conjugate Gradient and GMRES iteration methods after pre-conditioning the matrix so as to improve accuracy in computations. (Ref: 3, 4)

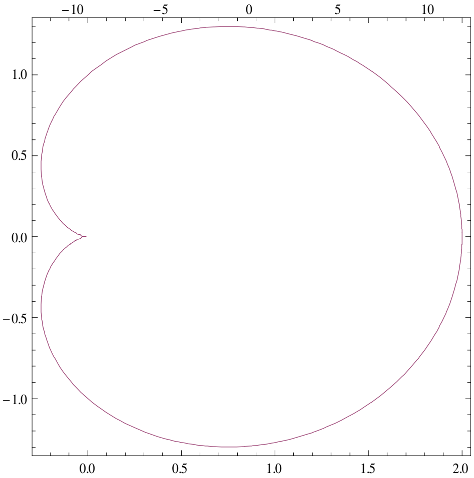
Data from circle (10 points on the curve), square (400 points on the curve), parabola (1001 points on the curve), Archimedean spiral (1000 points on the curve) and cardioid (1000 points on the curve) were fitted (in 2D) using $\phi(r) = r$ and by direct inversion of matrix B in Matlab. The results were as follows.





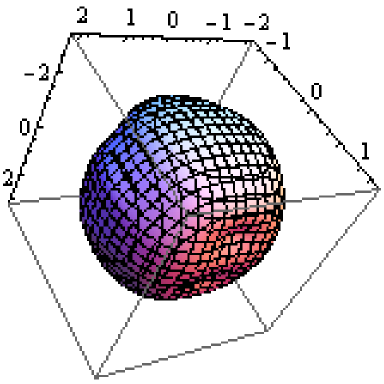


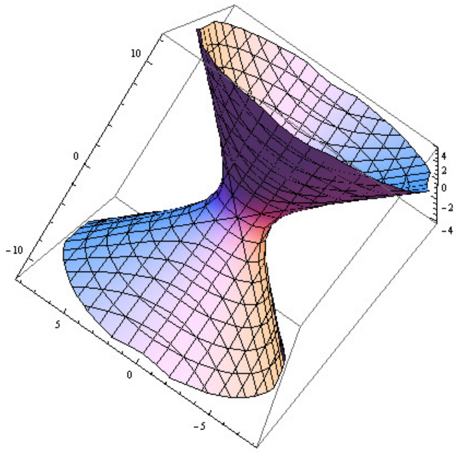


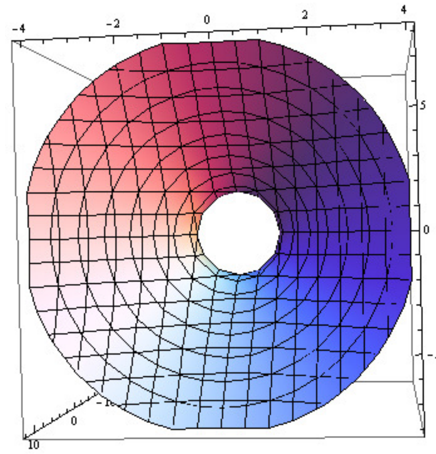


For the closed figures, viz. Circle, square and cardioids, the fits are pretty good. For the parabola, there are some stray loops in the figure, due to the property of the R.B.F.s that they try to produce closed figures. Similarly, the end-points of the spiral have been joined to produce a closed figure.

In 3D, data were fitted from ellipsoid and elliptic hyperboloid of one sheet. Here, the basic function used was $\phi(r) = r^2 \ln r$, the polynomial was quadratic and CG method was used for inversion. Pre-conditioning is also important in these cases (Ref: 3, 4).Besides, direct evaluation of the function-values is too memory-consuming in this case, so Fast Multipole Methods (F.M.M.) were used (Ref: 2). In these methods, no attempt is made to exactly evaluate the function values. Instead, they are evaluated only to a pre-assigned accuracy. This method is an order of magnitude faster than direct evaluation. The results were as follows.







Two views for the elliptic hyperboloid of one sheet.

Generalisation to nD:

In nD, with $\phi(r) = r^2 \ln r$ and a quadratic polynomial, the B matrix is still written as

$$B = \begin{bmatrix} A & P \\ P' & 0 \end{bmatrix},$$

but now $A_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ with $\|.\|$ being the Euclidean norm on $\mathbb{R}^n$, and i^{th} row of P = $[1, \alpha_{1i}, \alpha_{2i}, \ldots \alpha_{ni}, \alpha_{1i}^2, \alpha_{2i}^2, \ldots \alpha_{ni}^2]$ where $(\alpha_1, \alpha_2, \ldots \alpha_n)$ are the n coordinates of a point $\mathbf{x}$ in $\mathbb{R}^n$.

Solving this, we get the interpolant in the form $f(\mathbf{x}) = p(\mathbf{x}) + \Sigma^{N}_{i=1}\lambda_i\phi(||\mathbf{x}-\mathbf{x}_i||)$
where p is a quadratic polynomial in $\alpha_1, \alpha_2, \ldots \alpha_n$.

Acknowledgments:

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References:

1. J.C.Carr, R.K.Beatson, et. al., '**Reconstruction and Representation of 3D Objects with Radial Basis Functions**' (Siggraph 2001)

2. Rick Beatson and Leslie Greengard, '**A Short Course on Fast Multipole Methods**' (Wavelets, Multilevel methods and elliptic p.d.e., pages 1-37 , Oxford University Press 1997)

3. Jorge Nocedal and Stephen J. Wright, '**Numerical Optimization**' (Second Edition, Springer 2006)

4. R. K. Beatson, J. B. Cherrie, and C. T. Mouat, '**Fast fitting of radial basis functions: Methods based on preconditioned GMRES iteration**' Advances in Computational Mathematics, 11:253-270, 1999

5. Jonathan Richard Shewchuk, '**An Introduction to the Conjugate Gradient Method Without the Agonizing Pain**', August 4, 1994 - available online

Future Work:

In nD, we have to figure out a fast method to actually evaluate the function at various points. It is extremely difficult to get real-life problems for dimensions of 4 or higher, so it has not been possible as yet to test these methods for the higher dimensions. That is definitely another aim for the future.

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